Iterated Integrals 迭代积分(p966)

In Chapter 13, you saw that it is meaningful to differentiate functions of several variables with respect to one variable while holding the other variables constant. You can integrate functions of several variables by a similar procedure.

For example, for the multi-variable function $z = f(x, y)$, if the partial derivative is

$$\frac{\partial z}{\partial x} = 2xy$$

By considering y constant, you can integrate with respect to x to obtain

$$z = f(x, y) = \int \frac{\partial z}{\partial x} dx \quad \text{Intergrate with respect to x}$$

$$= \int 2xy\, dx \quad \text{Hold } y \text{ constant}$$

$$= y \int 2x\, dx \quad \text{Factor out constant y}$$

$$= y(x^2) + C(y) \quad \text{Antiderivative}$$

The "constant" of integration, $C(y)$, is a function of y. In other words, by integrating with respect to x, you are able to recover $f(x, y)$ only partially.

For now, you will focus on extending definite integrals to functions of several variables. For instance, by considering y constant, you can apply the Fundamental Theorem of Calculus to evaluate

$$\int_1^{2y} 2xy\, dx = x^2y\Big]_1^{2y} = (2y)^2y - (1)^2y = 4y^3 - y.$$

x is the variable of integration and y is fixed.

Replace x by the limits of integration.

The result is a function of y.

Similarly, you can integrate with respect to y by holding x fixed.

Example1: The Integral of an Integral(二次积分). Evaluate

$$\int_1^2 \left[\int_1^x (2x^2y^{-2} + 2y)dy\right] dx$$

Solution:

$$\int_1^x (2x^2y^{-2} + 2y)dy \quad \text{Integrate with respect to } y$$

$$= \left[\frac{-2x^2}{y} + y^2\right]_1^x = 3x^2 - 2x - 1$$

$$\int_1^2 [3x^2 - 2x - 1]dx \quad \text{Integrate with respect to } x$$

$$= [x^3 - x^2 - x]_1^2 = 2 - (-1) = 3$$

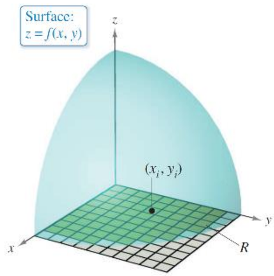
Iterated integrals – as shown in the example 1 - are usually written simply as

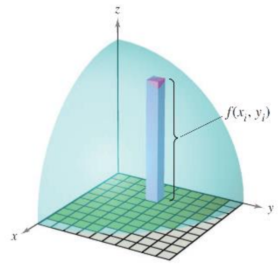
$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\ dx \quad \text{and} \quad \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\ dy$$

The **inside limits of integration** can be variable with respect to the outer variable of integration. However, the **outside limits of integration** must be constant with respect to both variables of integration.

Double Integrals 二重积分(p974)

Consider a continuous function $f(x, y) \geq 0$ such that for all (x, y) in a region R in the xy-plane. The goal is to find the volume of the solid region lying between the surface given by $z = f(x, y)$ and the xy-plane, as shown in right figure.





1. You can begin by superimposing a rectangular grid over the region.
2. The rectangles lying entirely within R form an inner partition Δ whose norm $\|\Delta\|$ is defined as the length of the longest diagonal of the rectangles.
3. choose a point (x_i, y_i) in each rectangle and form the rectangular prism whose height is $f(x_i, y_i)$ and the base has an area of ΔA_i. So that the volume of the ith prism is $f(x_i, y_i)\Delta A_i$
4. you can approximate the volume of the solid region by the Riemann sum of the volumes of all n prisms,

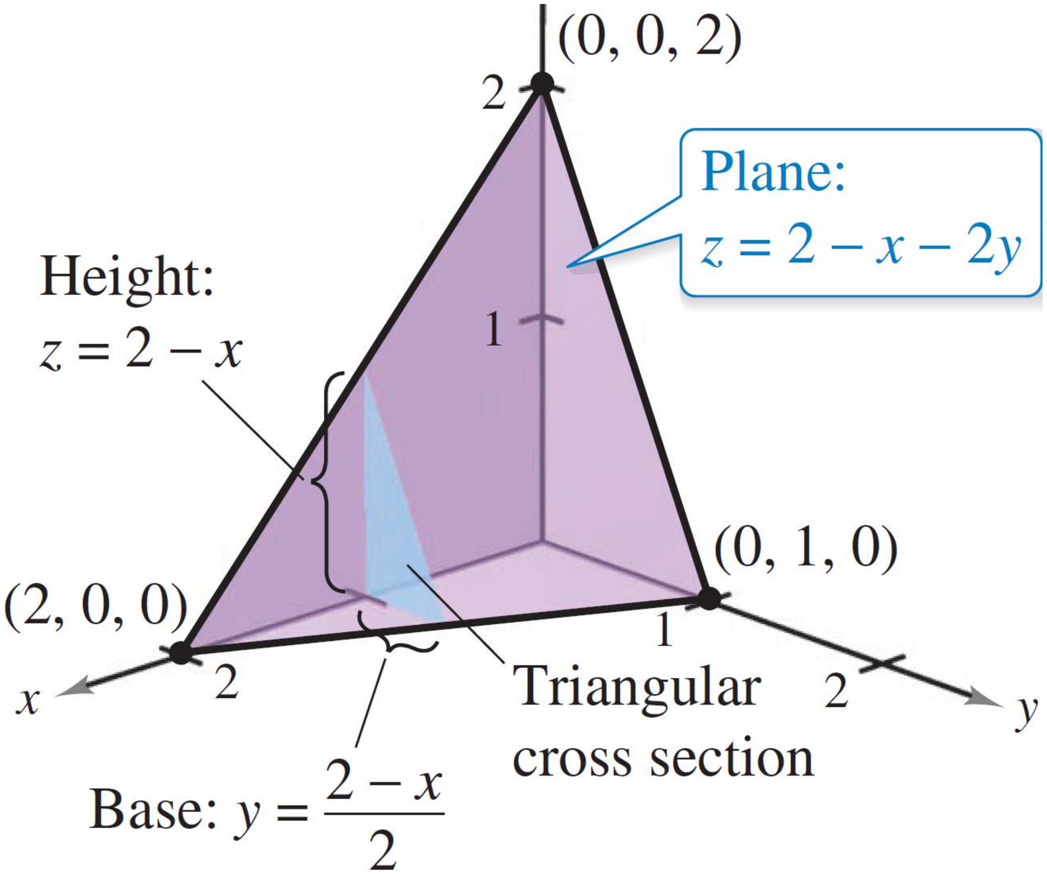
Double Integral 二重积分

If f is defined on a closed, bounded region R in the xy-plane, then the double integral of f over R is

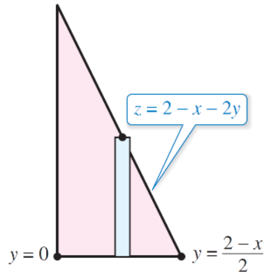
$$\iint_R f(x,y)\,dA = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$$

provided the limit exists. If the limit exists, then f is integrable over R.

Example2: Evaluation of Double Integrals. Evaluate the volume of the solid region bounded by the plane $z = f(x,y) = 2 - x - 2y$ and the three coordinate planes.



Solution 1: check the cross-section area A as a function of x: Each vertical cross section taken parallel to the yz-plane is a triangular region whose base has a length of $y = (2-x)/2$ and whose height is $z = 2 - x$. This implies that for a fixed value of x the area of the triangular cross section is



$$A(x) = \frac{1}{2}\left(\frac{2-x}{2}\right)(2-x) = \frac{(2-x)^2}{4}$$

The volume of the solid is

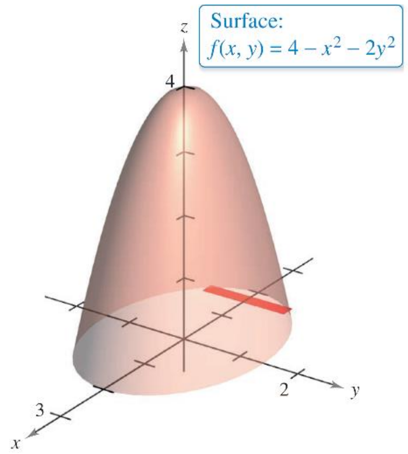
$$V = \int_a^b A(x)\,dx = \int_0^2 \frac{(2-x)^2}{4}\,dx = \left[-\frac{(2-x)^3}{12}\right]_0^2 = \frac{2}{3}$$

Solution 2: Double integral

$$V = \iint_R f(x,y)dA = \int_0^2 \int_0^{(2-x)/2} (2 - x - 2y)\,dy\,dx$$

x from 0 to 2 $\quad$ y from 0 to $(2-x)/2$

Example3: Find the volume of the solid region bounded by the paraboloid $z = 4 - x^2 - 2y^2$ and the xy-plane.



Solution 1: check the cross-section area A as a function of z

$$z = 4 - x^2 - 2y^2$$

$$\Rightarrow \frac{x^2}{4-z} + \frac{y^2}{(4-z)/2} = 1$$

$$\Rightarrow A(z) = (4-z)\pi/\sqrt{2}$$

$$V = \int_0^4 A(z)dz = \int_0^4 \frac{(4-z)\pi}{\sqrt{2}}\,dz$$

$$= \left(\frac{(4z - z^2/2)\pi}{\sqrt{2}}\right)_0^4 = 4\sqrt{2}\pi$$

Solution 2: $V = \iint_R f(x,y)dA = \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (4 - x^2 - 2y^2)\,dy\,dx$

Work and Energy 功和能：

Work 功：

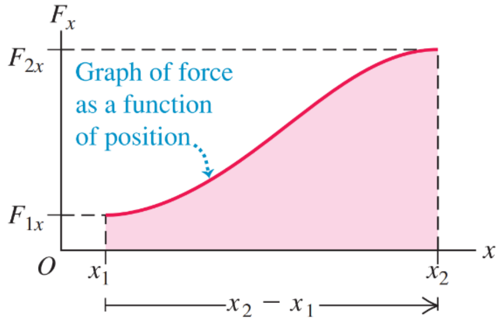
$\vec{F}$ is a constant:

$$W = \vec{F} \cdot \vec{s} = Fs\cos\theta$$

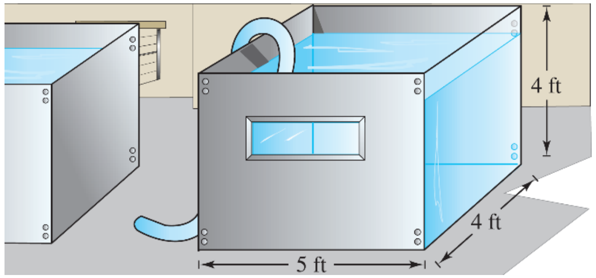
$\vec{F}$ is varying:

$$W = \lim_{\Delta x \to 0} \sum_{i=1}^{n} F_i \Delta x_i = \int_{x1}^{x2} F_x dx$$

(straight line displacement)



Example 4: A rectangular tank with a base 4 feet by 5 feet and a height of 4 feet is full of water. The water's density is ρ=62.4 pounds per cubic foot. How much work is done in pumping water out over the top edge in order to empty half of the tank?

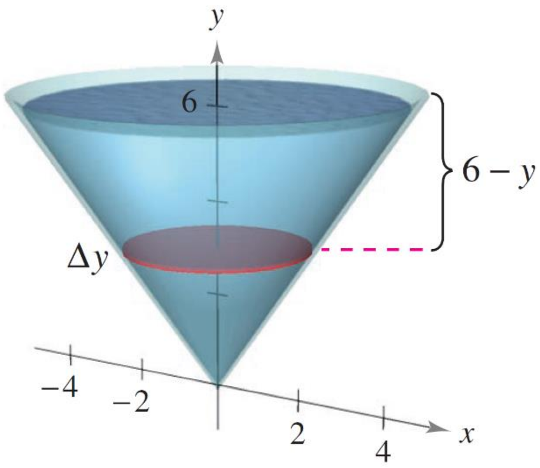


Water's mass: $m=\rho V,\ V=sx,\ s = 4 \cdot 5$ (square feet)

$$\text{Work: } W = \int_0^2 F dx = \int_0^2 mg dx = \int_0^2 \rho V dx = \int_0^2 \rho s x dx = \left(\rho s \frac{x^2}{2}\right)_0^2$$

$$= 62.4 \cdot 4 \cdot 5 \cdot \frac{4}{2} = 2496\ (feet \cdot pound)$$

Example 5: An open tank has the shape of a right circular cone. The tank is 8 meter across the top and 6 meter high. Suppose the density of water is ρ. How much work is done in emptying the tank by pumping the water over the top edge?



Slice a disk of the cone at the position y with height dy. The radius r of the disk is:

$$\frac{r}{y} = \frac{(8/2)}{6} \Rightarrow r = \frac{2}{3}y$$

The volume of the disk dV is:

$$dV = \pi r^2 dy = \pi\left(\frac{2}{3}y\right)^2 dy$$

The weight of the disk is:

$$dF = dmg = \rho g dV = \rho g\left(\frac{4}{9}\pi y^2 dy\right)$$

Distance to lift the disk of water out of the tank is:

$6 - y$

Work to be done on it is:

$$W = \int_0^6 (6-y) dF = \int_0^6 (6-y)\rho g \frac{4}{9}\pi(y^2 dy)$$

$$W = \frac{4}{9}\pi\rho g\left(2y^3 - \frac{1}{4}y^4\right)_0^6 = 48\pi\rho g\quad (J)$$

Coulomb's Law 库伦定律

$$F = \frac{k|q_1 q_2|}{r^2}$$

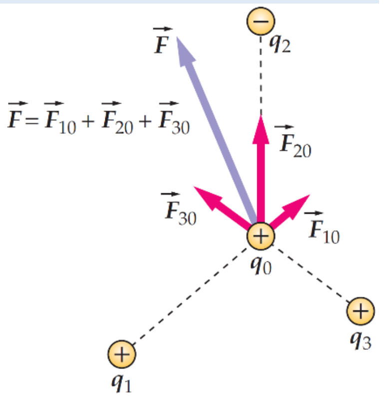
$$k = 8.99 \times 10^9\ N \cdot m^2/C^2$$

The Electric Field 电场

$$\vec{E} = \frac{\vec{F}}{q_0}\qquad (N/C)$$

$\vec{F}$: Net force on q_0.

The right figure shows an electric field of q_1, q_2, q_3 on q_0.

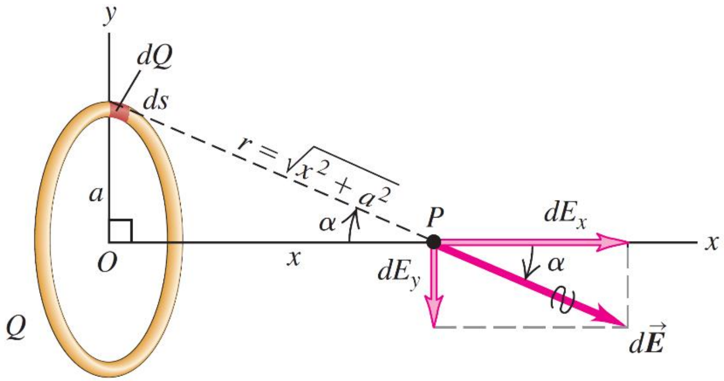


The magnitude of the electric field is also called the **electric field strength**.

The electric field at point P due to q_i charge is thus

$$\vec{E}_{iP} = \frac{kq_i}{r_{iP}^2}\vec{r}_{iP}\qquad \vec{E}_P = \sum \vec{E}_{iP}$$

Example 6: Charge Q^+ is uniformly distributed around a conducting ring of radius a. Find the electric field at a point P on the ring axis at a distance x from its center.



Solution:

λ is the linear charge density:

$\lambda = Q/2\pi a$

Divide the ring into infinitesimal segments ds, the charge in a segment of length ds is dQ

$dQ = \lambda ds$

As the ring is symmetry, the net field $\vec{E}$ at P is along the x-axis:

$\vec{E} = E_x\vec{\imath}$

The distance from the segment ds to the point is r:

$r^2 = x^2 + a^2$

$$d\vec{E} = \frac{kdQ}{r^2} = \frac{k\lambda ds}{x^2+a^2}$$

$$dE_x = dE\cos\alpha = \left(\frac{k\lambda ds}{x^2+a^2}\right)\left(\frac{x}{r}\right) = \frac{k\lambda x}{(x^2+a^2)^{3/2}}ds$$

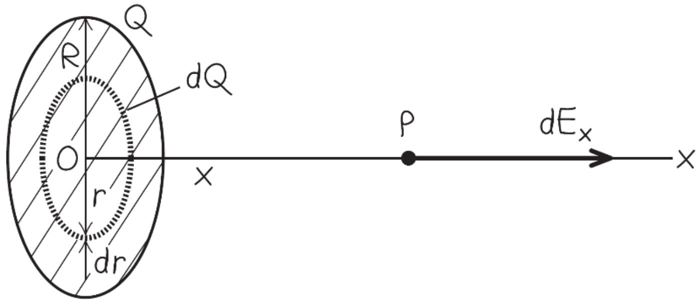
$$E_x = \int dE_x = \frac{k\lambda x}{(x^2+a^2)^{3/2}}\int_0^{2\pi a} ds = \frac{k\lambda x}{(x^2+a^2)^{3/2}}(2\pi a)$$

$$\vec{E} = E_x\vec{\imath} = \frac{2\pi a k\lambda x}{(x^2+a^2)^{3/2}}\vec{\imath} = \frac{kQx}{(x^2+a^2)^{3/2}}\vec{\imath}$$

Discussion:

$$\begin{cases} when\ x = 0 \ \rightarrow \ \vec{E} = 0 \\ when\ x \gg a \ \rightarrow \ \vec{E} = \dfrac{kQ}{x^2}\vec{\imath} \quad \text{(similar to a point charge)} \end{cases}$$

Example 7: A nonconducting disk of radius R has a uniform positive surface charge density σ. Find the electric field at a point along the axis of the disk a distance x from its center. Assume that x is positive.



Solution:

We can represent the charge distribution as a collection of concentric rings of Charge dQ . In Example 1 we obtained $E_x\vec{\imath}$ for the field on the axis of a single uniformly charged ring, so all we need do here is integrate the contributions of our rings.

A typical ring has inner radius r, and outer radius $r + dr$. The area:

$dA = 2\pi r dr$.

The charge dQ on this ring is:

$dQ = \sigma dA = 2\pi\sigma r dr$

We can use dQ in place of Q in the equation of example 1. Also use r in the place of a.

$$dE_x = \frac{kxdQ}{(x^2+r^2)^{3/2}} = \frac{kx(2\pi\sigma r dr)}{(x^2+r^2)^{3/2}}$$

Integrate the equation from $r = 0$ to $r = R$

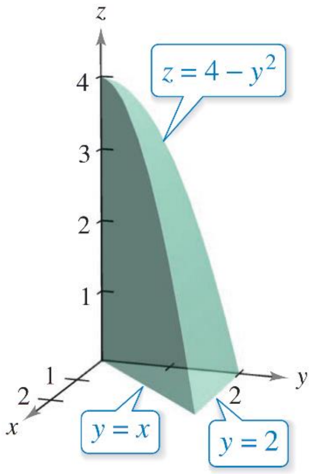
$$E_x = \int dE_x = \int_0^R \frac{\pi kx\sigma(2rdr)}{(x^2+r^2)^{3/2}}$$

let $u = x^2 + r^2, du = 2rdr$, when $r = R, u = x^2 + R^2$

$$E_x = \int_0^R \frac{\pi kx\sigma(2rdr)}{(x^2+r^2)^{\frac{3}{2}}} = \int_{x^2}^{x^2+R^2} \frac{\pi kx\sigma}{u^{\frac{3}{2}}}du = \pi kx\sigma\left(\frac{u^{-\frac{1}{2}}}{-\frac{1}{2}}\right)_{x^2}^{x^2+R^2}$$

$$E_x = \pi kx\sigma\left(\frac{u^{-\frac{1}{2}}}{-1/2}\right)_{x^2}^{x^2+R^2} = 2\pi kx\sigma\left(\frac{1}{\sqrt{u}}\right)_{x^2+R^2}^{x^2} = 2\pi kx\sigma\left(\frac{1}{x} - \frac{1}{\sqrt{x^2+R^2}}\right)$$

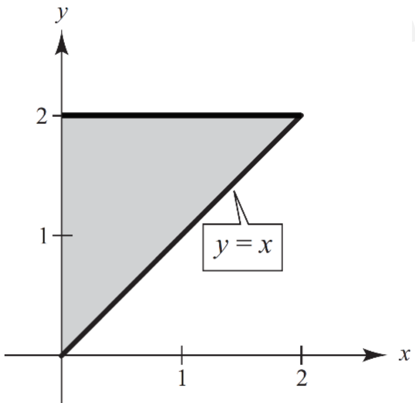
Exercise 1. Use a double integral to find the volume of the indicated solid .



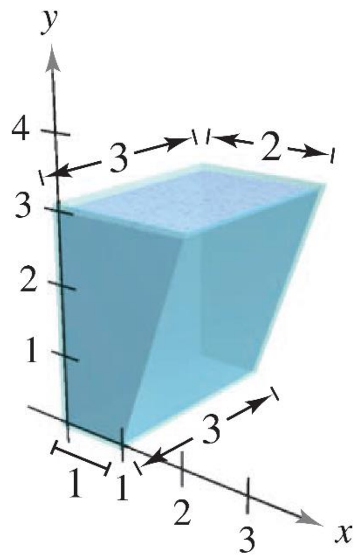
$$V = \int_0^2 \int_0^y (4 - y^2)dx\,dy$$

$$= \int_0^2 (4y - 3y^3)dy$$

$$= \left[2y^2 - \frac{y^4}{4}\right]_0^2 = 4$$



Exercise2. The fuel tank on a truck has trapezoidal cross sections with the dimensions (in feet) shown in the figure. Assume that the engine is approximately 3 feet above the top of the fuel tank and that diesel fuel weighs approximately 53.1 pounds per cubic foot. Find the work done by the fuel pump in raising a full tank of fuel to the level of the engine.



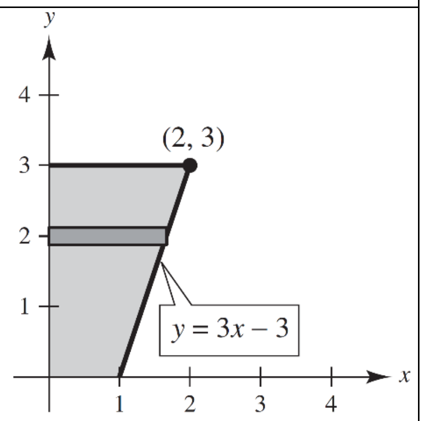
Volume of each layer: $\frac{y+3}{3}(3)\Delta y = (y + 3)\Delta y$

Weight of each layer: $53.1(y + 3)\Delta y$

Distance: $6 - y$

$$W = \int_0^3 53.1(y + 3)(6 - y)dy$$

$$= 53.1 \int_0^3 (18 + 3y - y^2)dy$$



$$= 53.1\left[18y + \frac{3y^2}{2} - \frac{y^3}{3}\right]_0^3 = 53.1\left(\frac{117}{2}\right) = 3106.35\ (ft - lb)$$

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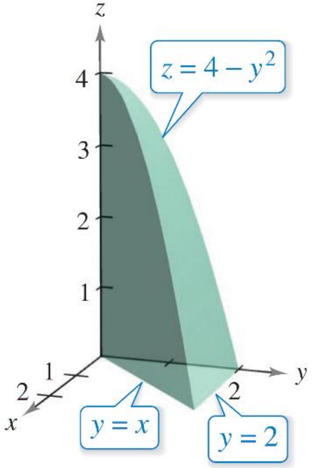

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